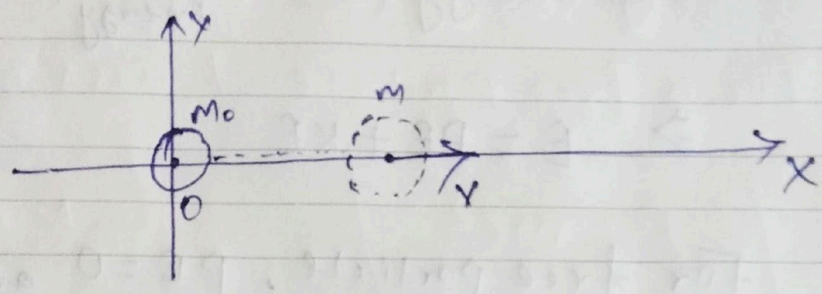
Derivation of $E=mc^2$



In Einstein's Special theory of relativity.

When any object is moving with velocity v
Then its relativistic mass is given by

$$m = \gamma m_0$$

where γ is Lorentz Factor
and m_0 is rest mass of the object.

According to Einstien's Special theory of relativity and Lorentz Transformation,

$$\text{Lorentz Factor } (\gamma) = \frac{1}{\sqrt{1 - v^2/c^2}}$$

and Relativistic momentum $p = \gamma m_0 v = mv$

From Mechanics, We know that

Total Energy (E) = Potential Energy (PE) + Kinetic Energy (KE)

$$\Rightarrow \quad E = PE + KE \qquad \text{—(i)}$$

For free particle, $PE = 0$ and $KE = \frac{1}{2}mv^2$

$$\Rightarrow \quad E = 0 + \frac{1}{2}mv^2 = \frac{1}{2}mv^2 \qquad \text{—(ii)}$$

According to Work-Energy Theorem, we know,

KE = work done

$$= \int F \cdot dx = \int \frac{dP}{dt} dx$$

$$= \int \frac{d(mv)}{dt} dx = \int \left(v\frac{dm}{dt} + m\frac{dv}{dt} \right) dx$$

$$= \int v\frac{dm}{dt} dx + m\frac{dv}{dt} dx = \int v\,dm \frac{dx}{dt} + m\,dv \frac{dx}{dt}$$

$$KE = \int v^2 dm + mv\,dv \qquad \text{—(iii)}$$

$m = \gamma m_0$

$\Rightarrow \quad m = \dfrac{m_0}{\sqrt{1 - V^2/c^2}} \Rightarrow m^2\left(1 - \dfrac{V^2}{c^2}\right) = m_0^2$

$\Rightarrow \quad m^2(c^2 - V^2) = m_0^2 c^2$

$\Rightarrow \quad m^2 c^2 - m^2 V^2 = m_0^2 c^2$

change into differential form

$\Rightarrow \quad c^2 2m\,dm - 2V^2 m\,dm - 2m^2 V\,dV = 0$

$\Rightarrow \quad c^2 dm = V^2 dm + mV\,dV \qquad \text{—(iv)}$

From eqn (iii) and (iv)

$$KE = \int_{m_0}^{m} c^2 dm = c^2(m - m_0)$$

$\Rightarrow \boxed{KE = (m - m_0)c^2} \qquad \text{—(v)}$

Here $m - m_0$ = change in mass $= \Delta m$

$\Rightarrow \quad E = (m - m_0)c^2 \qquad \text{—(vi)}$

Here Δm implies increase in Inertia of the Object.

Mass is the measure of Inertia.

If we generalise Δm by m, then

For a free particle of mass m,

$$\boxed{E = mc^2} \quad \text{— (vii)}$$

This is known as principle of Mass-Energy Equivalence.

This principle was proposed by Einstien in his original paper "Does the Inertia of a body depend upon its energy contents?" on

21 November 1905